Note

A note on the colorful fractional Helly theorem

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Minki Kim

Department of Mathematical Sciences, KAIST, Daejeon, South Korea

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ABSTRACT

Helly's theorem is a classical result concerning the intersection patterns of convex sets in $\mathbb{R}^d$. Two important generalizations are the colorful version and the fractional version. Recently, Bárány et al. combined the two, obtaining a colorful fractional Helly theorem. In this paper, we give an improved version of their result.

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1. Introduction

Helly's theorem is one of the most well-known and fundamental results in combinatorial geometry, which has various generalizations and applications. It was first proved by Helly [12] in 1913, but his proof was not published until 1923, after alternative proofs by Radon [17] and König [15]. We recommend the survey paper by Amenta, Loera, and Soberón [4] for an overview of previous results and open problems related to Helly's theorem. Recall that a family is *intersecting* if the intersection of all members is non-empty. The following is the original version of Helly's theorem.

Theorem 1.1 (*Helly's Theorem, Helly [12]*). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ with $|\mathcal{F}| \geq d + 1$. Suppose every $(d + 1)$-tuple of $\mathcal{F}$ is intersecting. Then the whole family $\mathcal{F}$ is intersecting.*

The following variant of Helly's theorem was found by Lovász, whose proof appeared first in a paper by Bárány [5]. Note that the original theorem by Helly is obtained by setting $\mathcal{F}_1 = \mathcal{F}_2 = \cdots = \mathcal{F}_{d+1}$.

Theorem 1.2 (*Colorful Helly Theorem, Lovász [5]*). *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite, non-empty families (color classes) of convex sets in $\mathbb{R}^d$ such that every colorful $(d + 1)$-tuple is intersecting. Then, for some $1 \leq i \leq d + 1$, the whole family $\mathcal{F}_i$ is intersecting.*

One way to generalize Helly's theorem is by weakening the assumption: not necessarily all but only a positive fraction of $(d + 1)$-tuples are intersecting. The following theorem shows how the conclusion changes.

Theorem 1.3 (*Fractional Helly Theorem, Katchalski and Liu [14]*). *For every $\alpha \in (0, 1]$, there exists $\beta = \beta(\alpha, d) \in (0, 1]$ such that the following holds: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ with $|\mathcal{F}| \geq d + 1$. If at least $\alpha \binom{|\mathcal{F}|}{d+1}$ of the $(d + 1)$-tuples in $\mathcal{F}$ are intersecting, then $\mathcal{F}$ contains an intersecting subfamily of size at least $\beta|\mathcal{F}|$.*

E-mail address: kmk90@kaist.ac.kr.

The fractional variant of Helly's theorem first appeared as a conjecture on interval graphs, i.e. intersection graphs of families of intervals on $\mathbb{R}$. Abbott and Katchalski [1] proved that $\beta = 1 - \sqrt{1-\alpha}$ is optimal for every family whose intersection graph is a chordal graph. Note that, by a result of Gavril [10], interval graphs are chordal graphs.

The fractional Helly theorem for arbitrary dimensions was proved by Katchalski and Liu [14]. Their proof gives a lower bound $\beta \geq \alpha/(d+1)$, and also shows that β tends to 1 as α tends to 1. Note that the original theorem by Helly is obtained by setting $\alpha = 1$. Later, the quantitatively sharp value $\beta(\alpha, d) = 1 - (1-\alpha)^{1/(d+1)}$ was found by Kalai [13] and Eckhoff [7], which is a consequence of the upper bound theorem for families of convex sets.

The (p, q)-theorem, another important generalization of Helly's theorem, deals with a weaker version of the assumption, the so-called (p, q)-condition: for every p members in a given family, there are some q members of the family that are intersecting. For instance, the $(d+1, d+1)$-condition in $\mathbb{R}^d$ is the hypothesis of Helly's theorem. The (p, q)-theorem was proved by Alon and Kleitman [3], settling a conjecture by Hadwiger and Debrunner [11]. It states as follows.

Theorem 1.4 ((p, q)-Theorem, Alon and Kleitman [3]). *Let p, q and d be integers with $p \geq q \geq d+1$. Then there exists a number $HD_d(p, q)$ such that the following is true: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-condition. Then $\mathcal{F}$ has a transversal consisting of at most $HD_d(p, q)$ points.*

The original proof of the (p, q)-theorem is quite long and involved, using various techniques. It was later shown by Alon et al. [2] that the most crucial ingredient is the fractional Helly theorem, and they showed that one can obtain a (p, q)-theorem for abstract set-systems which satisfy an appropriate "fractional Helly property". For an overview and further knowledge of this field, see the survey papers by Eckhoff [8,9] and the textbook by Matoušek [16].

Recently, Bárány et al. [6] established colorful and fractional versions of the (p, q)-theorem. A key ingredient in their proof was a colorful variant of the fractional Helly theorem.

Theorem 1.5 (Bárány, Fodor, Montejano, Oliveros, and Pór [6]). *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite, non-empty families (color classes) of convex sets in $\mathbb{R}^d$, and assume that $\alpha \in (0, 1]$. If at least $\alpha |\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d+1)$-tuples are intersecting, then some $\mathcal{F}_i$ contains an intersecting subfamily of size $\frac{\alpha}{d+1} |\mathcal{F}_i|$.*

The proof in [6] follows the standard argument where each intersecting colorful $(d+1)$-tuple is charged to one of its d-tuples. (See for instance section 8.1 in [16] for a proof of the uncolored version.)

Note that for $\alpha = 1$ we recover the hypothesis of the colorful Helly theorem, and it is natural to ask whether the function β tends to 1 as α tends to 1. This problem is implicitly contained in the paper by Bárány et al. [6] and was communicated to us by F. Fodor.

Here we solve this problem by showing the following.

Theorem 1.6. *For every $\alpha \in (0, 1]$, there exists $\beta = \beta(\alpha, d) \in (0, 1]$ tending to 1 as α tends to 1 such that the following holds: Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite, non-empty families (color classes) of convex sets in $\mathbb{R}^d$. If at least $\alpha |\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d+1)$-tuples are intersecting, then for some $1 \leq i \leq d+1$, $\mathcal{F}_i$ contains an intersecting subfamily of size $\beta |\mathcal{F}_i|$.*

In order to prove Theorem 1.6, we will show that for every sufficiently small $\epsilon > 0$, if none of the $\mathcal{F}_i$ have an intersecting subfamily of size $(1-\epsilon)|\mathcal{F}_i|$, then there is a *positive fraction* of the colorful $(d+1)$-tuples which are *non-intersecting*. This will be done with explicit calculations.

An interesting aspect of our proof is that it is purely combinatorial (formulated in the language of uniform hypergraphs) and uses only the colorful Helly theorem as a "black box". Our method can easily be modified to provide another (simple) proof that the function β tends to 1 as α tends to 1 in the classical fractional Helly theorem (Theorem 1.3), but it does not give the optimal bound of Kalai and Eckhoff.

2. Proof of Theorem 1.6

2.1. The matching number of hypergraphs

Let $\mathcal{H}$ be an r-uniform hypergraph on a vertex set X. A subset $S \subseteq X$ is said to be an *independent set* in $\mathcal{H}$ if the induced sub-hypergraph $\mathcal{H}[S]$ contains no hyperedge. The *independence number* $\alpha(\mathcal{H})$ of $\mathcal{H}$ is the cardinality of a maximum independent set in $\mathcal{H}$. A *matching* of $\mathcal{H}$ is a set of pairwise disjoint edges in $\mathcal{H}$. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the cardinality of a maximum matching in $\mathcal{H}$. We need the following observation.

Observation 2.1. *Let $\mathcal{H} = (X, E)$ be an r-uniform hypergraph with $|X| = n$. Suppose*

$$\alpha(\mathcal{H}) < cn$$

for some $c \in (0, 1]$. Let $M = \{e_1, \ldots, e_\nu\}$ be a maximum matching in $\mathcal{H}$. Note that $X \setminus (e_1 \cup \cdots \cup e_\nu)$ is an independent set in $\mathcal{H}$. If not, assume that there is an edge e contained in $X \setminus (e_1 \cup \cdots \cup e_\nu)$. Then $M \cup \{e\}$ is a matching in $\mathcal{H}$, which is a contradiction to the maximality of M. Thus

$$|X \setminus (e_1 \cup \cdots \cup e_\nu)| = n - r\nu(\mathcal{H}) \leq \alpha(\mathcal{H}) < cn,$$

so $\nu(\mathcal{H}) > \frac{n - cn}{r}$.

2.2. Proof of Theorem 1.6

We will show the following more explicit result.

Theorem 2.2. *For every $\alpha \in (0, 1]$, the following holds: Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite families (color classes) of convex sets in $\mathbb{R}^d$. If at least $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d + 1)$-tuples are intersecting, then for some $1 \leq i \leq d + 1$, $\mathcal{F}_i$ contains an intersecting subfamily of size at least*

$$\max \left\{ \frac{\alpha}{d + 1}, 1 - (d + 1)(1 - \alpha)^{\frac{1}{d+1}} \right\} |\mathcal{F}_i|.$$

The following is a key lemma for the proof of Theorem 2.2.

Lemma 2.3. *Choose any subfamily from each color class, say $\mathcal{F}_1', \ldots, \mathcal{F}_{d+1}'$. If each of $\mathcal{F}_i'$ is not intersecting, then at least one of colorful $(d + 1)$-tuple is not intersecting.*

Proof. This follows directly from the colorful Helly theorem. □

Proof of Theorem 2.2. It is sufficient to show that for every $\alpha \in \left[1 - \frac{1}{(d+1)^{(d+1)}}, 1 \right]$, if at least $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d+1)$-tuples are intersecting, then some $\mathcal{F}_i$ contains an intersecting subfamily of size at least $\left(1 - (d + 1)(1 - \alpha)^{\frac{1}{d+1}} \right) |\mathcal{F}_i|$.

Let $\mathcal{F}$ be the disjoint union of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$. For each $1 \leq i \leq d + 1$, let $n_i = |\mathcal{F}_i|$ and define a $(d + 1)$-uniform hypergraph $H_i := (\mathcal{F}_i, E_i)$ whose hyperedges are the *non-intersecting* $(d + 1)$-tuples in $\mathcal{F}_i$. Let $\nu_i = \nu(H_i)$ for each $1 \leq i \leq d + 1$.

Also define a $(d + 1)$-uniform hypergraph $H := (\mathcal{F}, E)$ whose hyperedges are the *intersecting* colorful $(d + 1)$-tuples in $\mathcal{F}$.

Given $\alpha \in \left[1 - \frac{1}{(d+1)^{(d+1)}}, 1 \right]$, let $\gamma = \gamma(\alpha, d) = 1 - (d + 1)(1 - \alpha)^{\frac{1}{d+1}}$. To show a contradiction, assume that in each family $\mathcal{F}_j$, every subfamily of size at least γn_j has an empty intersection.

By our hypothesis we have

$$\alpha n_1 \cdots n_{d+1} \leq |E|,$$

and by Lemma 2.3 we have

$$|E| \leq n_1 \cdots n_{d+1} - \nu_1 \cdots \nu_{d+1}.$$

Combining this with Observation 2.1, $\nu_j > \frac{n_j - \gamma n_j}{d+1} = \left(\frac{1 - \gamma}{d + 1} \right) n_j$, we obtain

$$\begin{aligned}
\alpha n_1 \cdots n_{d+1} &\leq n_1 \cdots n_{d+1} - \nu_1 \cdots \nu_{d+1} \\
&< n_1 \cdots n_{d+1} - \left(\frac{1 - \gamma}{d + 1} \right)^{d+1} n_1 \cdots n_{d+1} \\
&= \left(1 - \left(\frac{1 - \gamma}{d + 1} \right)^{d+1} \right) n_1 \cdots n_{d+1},
\end{aligned}$$

hence $\alpha < 1 - (\frac{1-\gamma}{d+1})^{d+1} = \alpha$, which is a contradiction.

Thus, there should exist $1 \leq i \leq d+1$ such that $\mathcal{F}_i$ contains an intersecting subfamily of size $(1 - (d + 1)(1 - \alpha)^{\frac{1}{d+1}})n_i$. □

3. The upper bound

In this section, we prove the following.

Theorem 3.1. *For every $\alpha \in (0, 1]$, there exist finite families (color classes) $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of convex sets in $\mathbb{R}^d$ such that the following holds. $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d+1)$-tuples are intersecting, but in each color class $\mathcal{F}_i$, the maximum cardinality of an intersecting subfamily is at most $\left(1 - (1 - \alpha)^{\frac{1}{d+1}} \right) |\mathcal{F}_i|$.*

First recall that in the fractional Helly theorem, the upper bound is given by

$$\beta = \beta(\alpha, d) \leq \left(1 - (1 - \alpha)^{\frac{1}{d+1}} \right).$$

This can be seen by the following well-known construction, which also shows the exactness of upper bound theorem for convex sets [7,13].

Example 3.2. Let $\mathcal{F}$ consist of $\lfloor \beta n \rfloor - (d+1)$ copies of $\mathbb{R}^d$ and $n - \lfloor \beta n \rfloor + (d+1)$ hyperplanes in general position. Denote by $f_d(\mathcal{F})$ the number of intersecting $(d+1)$-tuples in $\mathcal{F}$. Note that

$$\alpha \binom{n}{d+1} = f_d(\mathcal{F}) = \binom{n}{d+1} - \binom{n - (\lfloor \beta n \rfloor - (d+1))}{d+1}.$$

The following example, which is conjectured to have the maximum number of intersecting $(d+1)$-tuples, shows Theorem 3.1.

Example 3.3. Let $\{v_1, \ldots, v_{d+1}\}$ form a (regular) d-simplex centered at the origin in $\mathbb{R}^d$. Let l_i be a half-line starting from the origin which passes through v_i. Define $R_i := \operatorname{conv}(\{l_1, \ldots, l_{d+1}\} \setminus \{l_i\})$. Let $K = K_1 \cup \cdots \cup K_{d+1}$ be a family of $\sum_{i=1}^{d+1} (n_i - \lfloor \beta n_i \rfloor + 1)$ hyperplanes in general position in $\mathbb{R}^d$ such that the following holds: for each $i = 1, \ldots, d+1$, $|K_i| = n_i - \lfloor \beta n_i \rfloor + 1$ and every member in K_i does not meet R_i. Note that each hyperplane in K_i meets l_i, and any colorful $(d+1)$-tuple in $K_1, \ldots, K_{d+1}$ is not intersecting.

Now, for each $i = 1, \ldots, d+1$, let $\mathcal{F}_i$ consist of $\lfloor \beta n_i \rfloor - 1$ copies of R_i and all the hyperplanes in K_i. Note that every intersecting subfamily in $\mathcal{F}_i$ has size at most βn_i. It is easy to see that every colorful $(d+1)$-tuple in $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ is intersecting, except the colorful $(d+1)$-tuples in $K_1, \ldots, K_{d+1}$.

Denote by $f_d(\mathcal{F}_1, \ldots, \mathcal{F}_{d+1})$ the number of intersecting colorful $(d+1)$-tuples of $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$. Then we have

$$\alpha n_1 \cdots n_{d+1} = f_d(\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}) = n_1 \cdots n_{d+1} - (n_1 - \lfloor \beta n_1 \rfloor + 1) \cdots (n_{d+1} - \lfloor \beta n_{d+1} \rfloor + 1).$$

4. Remarks

In this note, we found a lower bound on the function $\beta(\alpha, d) \geq 1 - (d+1)(1-\alpha)^{\frac{1}{d+1}}$ for $\alpha \in \left(1 - \left(\frac{1}{d+1}\right)^{d+1}\right]$, and an upper bound $\beta(\alpha, d) \leq 1 - (1-\alpha)^{\frac{1}{d+1}}$ in the colorful fractional Helly theorem.

It would be interesting to determine the exact value of $\beta(\alpha, d)$.

Problem 4.1. What is the exact value of $\beta = \beta(\alpha, d)$ in Theorem 1.6?

It is easy to see that $\beta(\alpha, 1) = 1 - \sqrt{1-\alpha}$ is the optimal bound for $d = 1$. We conjecture that $\beta(\alpha, d) = 1 - (1-\alpha)^{\frac{1}{d+1}}$ is the optimal bound for $d > 1$. This can be shown by proving the following.

Conjecture 4.2. *Let $n_i \geq k_i$ be given for $i = 1, \ldots, d+1$. Let $\mathcal{F}_1, \ldots \mathcal{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$ such that the following holds. For each $i = 1, \ldots, d+1$, $|\mathcal{F}_i| = n_i$ and there is no intersecting subfamily of size k_i in $\mathcal{F}_i$. Then the number of intersecting colorful $(d+1)$-tuples is at most*

$$n_1 \cdots n_{d+1} - (n_1 - k_1 + 1) \cdots (n_{d+1} - k_{d+1} + 1).$$

Acknowledgments

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